Gram-Schmidt orthogonalization

System of linear equations

We consider a system of linear equations of the form $\mathbf{A}{\cdot}\mathbf{x} = \mathbf{y}$, where the coefficient matrix has the form

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 4 \end{pmatrix}$$

and the right-hand side is given by the column vector

$$\mathbf{y} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Solution par orthogonalization Gram-Schmidt

Construct an orthogonal matrix **Q**, with $\boldsymbol{Q}^T \cdot \boldsymbol{Q} = \mathbf{1}$, such that

$$\mathbf{Q} = \mathbf{A} \cdot \mathbf{R},$$

where R is *upper triangular*.

Start from the three column vectors of **A**,

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix}$$

which are not orthogonal,

$$\mathbf{A}^{\mathrm{T}} \cdot \mathbf{A} = \begin{pmatrix} 14 & 20 & 23 \\ 20 & 29 & 34 \\ 23 & 34 & 41 \end{pmatrix} \neq \mathbf{1}.$$

Step 1

Take $\mathbf{u}_1 = \mathbf{a}_1$ as first unnormalized column vector of a basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ fulfilling $\boldsymbol{u}_i^T \cdot \boldsymbol{u}_j = |\boldsymbol{u}_i| |\boldsymbol{u}_j| \, \delta_{ij}$,

$$\mathbf{u}_1 = \mathbf{a}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Normalize this column vector and write

$$\mathbf{q}_1 = \begin{pmatrix} \frac{1}{\sqrt{14}} \\ \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{14}} \end{pmatrix} = R_{11}\mathbf{a}_1 \text{ , where } R_{11} = \frac{1}{|\mathbf{u}_1|} = \frac{1}{\sqrt{14}}$$

Step 2

Construct the second unnormalized basis vector

$$\mathbf{u}_2 = \mathbf{a}_2 - (\mathbf{q}_1^T \cdot \mathbf{a}_2)\,\mathbf{q}_1 = \begin{pmatrix} \frac{4}{7} \\ \frac{1}{7} \\ -\frac{2}{7} \end{pmatrix}$$

and normalize,

$$\mathbf{q}_2 = \begin{pmatrix} \frac{4}{\sqrt{21}} \\ \frac{1}{\sqrt{21}} \\ -\frac{2}{\sqrt{21}} \end{pmatrix}$$

Write

$$\mathbf{q}_2 = \frac{1}{|\mathbf{u}_2|} \left(\mathbf{a}_2 - \left(\mathbf{q}_1^T \cdot \mathbf{a}_2\right) \mathbf{q}_1\right) = \frac{1}{|\mathbf{u}_2|} \left(\mathbf{a}_2 - \left(\mathbf{q}_1^T \cdot \mathbf{a}_2\right) (R_{11}\,\mathbf{a}_1)\right) = R_{12}\,\mathbf{a}_1 + R_{22}\,\mathbf{a}_2$$

where

$$R_{12} = -\frac{10}{\sqrt{21}},\quad R_{22} = \sqrt{\frac{7}{3}}$$

Step 3

Construct the third and last unnormalized basis vector

$$\mathbf{u}_3 = \mathbf{a}_3 - (\mathbf{q}_1^T \cdot \mathbf{a}_3)\,\mathbf{q}_1 - (\mathbf{q}_2^T \cdot \mathbf{a}_3)\,\mathbf{q}_2 = \begin{pmatrix} -\frac{1}{6} \\ \frac{1}{3} \\ -\frac{1}{6} \end{pmatrix}$$

and normalize,

$$q_3 = \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix}.$$

Write

$$\mathbf{q}_3 = \frac{1}{|\mathbf{u}_3|} \left(\mathbf{a}_3 - \left(\mathbf{q}_1^T \cdot \mathbf{a}_3\right) \mathbf{q}_1 - \left(\mathbf{q}_2^T \cdot \mathbf{a}_3\right) \mathbf{q}_2\right) =$$

$$\frac{1}{|\mathbf{u}_3|} \left(\mathbf{a}_3 - \left(\mathbf{q}_1^T \cdot \mathbf{a}_3\right) (R_{11}\, \mathbf{a}_1) - \left(\mathbf{q}_2^T \cdot \mathbf{a}_3\right) (R_{12}\, \mathbf{a}_1 + R_{22}\, \mathbf{a}_2)\right) = R_{13}\, \mathbf{a}_1 + R_{23}\, \mathbf{a}_2 + R_{33}\, \mathbf{a}_3.$$

where

$$R_{13} = \frac{13}{\sqrt{6}}, \quad R_{23} = -8\sqrt{\frac{2}{3}}, \quad R_{33} = \sqrt{6}$$

This the final matrix **Q**,

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{14}} & \frac{4}{\sqrt{21}} & -\frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{7}} & \frac{1}{\sqrt{21}} & \sqrt{\frac{2}{3}} \\ \frac{3}{\sqrt{14}} & -\frac{2}{\sqrt{21}} & -\frac{1}{\sqrt{6}} \end{pmatrix},$$

and this the final matrix **R**,

$$\mathbf{R} = \begin{pmatrix} \frac{1}{\sqrt{14}} & -\frac{10}{\sqrt{21}} & \frac{13}{\sqrt{6}} \\ 0 & \sqrt{\frac{7}{3}} & -8\sqrt{\frac{2}{3}} \\ 0 & 0 & \sqrt{6} \end{pmatrix}.$$

Check that $\mathbf{Q} = \mathbf{A} \cdot \mathbf{R}$ et and that $\boldsymbol{Q}^T \cdot \boldsymbol{Q} = \boldsymbol{Q} \cdot \boldsymbol{Q}^T = 1$:

$$\mathbf{Q} - \mathbf{A} \cdot \mathbf{R} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\mathbf{Q}^T \cdot \mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\mathbf{Q} \cdot \mathbf{Q}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Inverse of **A** and solution of **A**·**x** = **y**

Since **Q** = **A**·**R** and $Q \cdot Q^T = 1$ it follows that $A \cdot R \cdot Q^T = \mathbf{1}$, i.e. that

$$A^{-1} = R \cdot Q^T.$$

Check

$$\mathbf{A} \cdot \mathbf{R} \cdot \mathbf{Q}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solve

$$\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{y} = \mathbf{R} \cdot \mathbf{Q}^T \cdot \mathbf{y} = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}.$$

Check

$$\mathbf{A} \cdot \mathbf{x} - \mathbf{y} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$