Definitions

Eigenvectors via Gauss elimination

Example 1

Eigenvalues

In[6]:=
```
Clear[Amat]
```

In[7]:=
```
MatrixForm[Amat = A[[1]]]
```

Out[7]//MatrixForm=

$$\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$$

In[8]:=
```
CharacteristicPolynomial[Amat, λ] // Factor
```

Out[8]= $(-6+\lambda)\ (-1+\lambda)$

In[9]:=
```
eigenvaluesAmat = Eigenvalues[Amat]
```

Out[9]= {6, 1}

In[10]:=
```
MatrixForm[Bmat[λ_] = Amat - λ * IdentityMatrix[2]]
```

Out[10]//MatrixForm=

$$\begin{pmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{pmatrix}$$

Eigenvector for eigenvalue 1

In[11]:=
```
MatrixForm[Bmat[eigenvaluesAmat[[1]]]]
```

Out[11]//MatrixForm=

$$\begin{pmatrix} -1 & 4 \\ 1 & -4 \end{pmatrix}$$

In[12]:=
```
luB1 = LUDecomposition[Bmat[eigenvaluesAmat[[1]]]] // Quiet
```

Out[12]=

{{{-1, 4}, {-1, 0}}, {1, 2}, 0}

In[13]:=
```
MatrixForm[lB1 = LowerTriangularize[luB1[[1]], -1] + IdentityMatrix[2]];
```

In[14]:=
```
MatrixForm[uB1 = UpperTriangularize[luB1[[1]]]]
```

Out[14]//MatrixForm=

$$\begin{pmatrix} -1 & 4 \\ 0 & 0 \end{pmatrix}$$

In[15]:=
```
sol1 = (Solve[uB1.{X, α} == {0, 0}, {X}][[1]]) // Quiet
```

Out[15]=

$\{X \to 4\,\alpha\}$

In[16]:=
```
evec1[α_] = {X, α} /. sol1
```

Out[16]=

$\{4\,\alpha,\ \alpha\}$

In[17]:=
```
ev1 = Normalize[evec1[1]]
```
Out[17]=

$$\left\{\frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}}\right\}$$

Eigenvector for eigenvalue 2

In[18]:=
```
MatrixForm[Bmat[eigenvaluesAmat[[2]]]]
```
Out[18]//MatrixForm=

$$\begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix}$$

In[19]:=
```
luB2 = LUDecomposition[Bmat[eigenvaluesAmat[[2]]]] // Quiet
```
Out[19]=

{{{1, 1}, {4, 0}}, {2, 1}, 0}

In[20]:=
```
MatrixForm[lB2 = LowerTriangularize[luB2[[1]], -1] + IdentityMatrix[2]];
```
In[21]:=
```
MatrixForm[uB2 = UpperTriangularize[luB2[[1]]]]
```
Out[21]//MatrixForm=

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

In[22]:=
```
sol2 = (Solve[uB2.{X, α} == {0, 0}, {X}][[1]]) // Quiet
```
Out[22]=

$\{X \to -\alpha\}$

In[23]:=
```
evec2[α_] = {X, α} /. sol2
```
Out[23]=

$\{-\alpha, \alpha\}$

In[24]:=
```
ev2 = Normalize[evec2[2]]
```
Out[24]=

$$\left\{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$$

Eigenbasis

In[25]:=
```
MatrixForm[Ebasis = Transpose[{ev1, ev2}]]
```
Out[25]//MatrixForm=

$$\begin{pmatrix} \frac{4}{\sqrt{17}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{17}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

In[26]:=
```
MatrixForm[G = Transpose[Ebasis].Ebasis // FullSimplify]
```
Out[26]//MatrixForm=

$$\begin{pmatrix} 1 & -\frac{3}{\sqrt{34}} \\ -\frac{3}{\sqrt{34}} & 1 \end{pmatrix}$$

Example 2

Eigenvalues

In[27]:=
```
Clear[Amat]
```

In[28]:=
```
MatrixForm[Amat = A⟦2⟧]
```

Out[28]//MatrixForm=

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

In[29]:=
```
CharacteristicPolynomial[Amat, λ] // Factor
```

Out[29]=

$(-3 + \lambda)\ (1 + \lambda)$

In[30]:=
```
eigenvaluesAmat = Eigenvalues[Amat]
```

Out[30]=

{3, -1}

In[31]:=
```
MatrixForm[Bmat[λ_] = Amat - λ * IdentityMatrix[2]]
```

Out[31]//MatrixForm=

$$\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix}$$

Eigenvector for eigenvalue 1

In[32]:=
```
MatrixForm[Bmat[eigenvaluesAmat⟦1⟧]]
```

Out[32]//MatrixForm=

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$$

In[33]:=
```
luB1 = LUDecomposition[Bmat[eigenvaluesAmat⟦1⟧]] // Quiet
```

Out[33]=

{{{-2, 2}, {-1, 0}}, {1, 2}, 0}

In[34]:=
```
MatrixForm[lB1 = LowerTriangularize[luB1⟦1⟧, -1] + IdentityMatrix[2]];
```

In[35]:=
```
MatrixForm[uB1 = UpperTriangularize[luB1⟦1⟧]]
```

Out[35]//MatrixForm=

$$\begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}$$

In[36]:=
```
sol1 = (Solve[uB1.{X, α} == {0, 0}, {X}]⟦1⟧) // Quiet
```

Out[36]=

{X → α}

In[37]:=
```
evec1[α_] = {X, α} /. sol1
```

Out[37]=

{α, α}

In[38]:=
```
ev1 = Normalize[evec1[1]]
```
Out[38]=

$$\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$$

Eigenvector for eigenvalue 2

In[39]:=
```
MatrixForm[Bmat[eigenvaluesAmat[[2]]]]
```
Out[39]//MatrixForm=

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

In[40]:=
```
luB2 = LUDecomposition[Bmat[eigenvaluesAmat[[2]]]] // Quiet
```
Out[40]=
```
{{{2, 2}, {1, 0}}, {1, 2}, 0}
```

In[41]:=
```
MatrixForm[lB2 = LowerTriangularize[luB2[[1]], -1] + IdentityMatrix[2]];
```

In[42]:=
```
MatrixForm[uB2 = UpperTriangularize[luB2[[1]]]]
```
Out[42]//MatrixForm=

$$\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$$

In[43]:=
```
sol2 = (Solve[uB2.{X, α} == {0, 0}, {X}][[1]]) // Quiet
```
Out[43]=

$\{X \to -\alpha\}$

In[44]:=
```
evec2[α_] = {X, α} /. sol2
```
Out[44]=

$\{-\alpha, \alpha\}$

In[45]:=
```
ev2 = Normalize[evec2[1]]
```
Out[45]=

$$\left\{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$$

Eigenbasis

In[46]:=
```
MatrixForm[Ebasis = Transpose[{ev1, ev2}]]
```
Out[46]//MatrixForm=

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

In[47]:=
```
MatrixForm[G = Transpose[Ebasis].Ebasis // FullSimplify]
```
Out[47]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example 3

Eigenvalues

In[48]:=
```
Clear[Amat]
```

In[49]:=
```
MatrixForm[Amat = A[[3]]]
```

Out[49]//MatrixForm=

$$\begin{pmatrix} -2 & 6 & -3 \\ -4 & 9 & -4 \\ -4 & 8 & -3 \end{pmatrix}$$

In[50]:=
```
CharacteristicPolynomial[Amat, λ] // Factor
```

Out[50]=

$$-\left((-2+\lambda)\ (-1+\lambda)^2\right)$$

In[51]:=
```
eigenvaluesAmat = Eigenvalues[Amat]
```

Out[51]=

{2, 1, 1}

In[52]:=
```
MatrixForm[Bmat[λ_] = Amat - λ * IdentityMatrix[3]]
```

Out[52]//MatrixForm=

$$\begin{pmatrix} -2-\lambda & 6 & -3 \\ -4 & 9-\lambda & -4 \\ -4 & 8 & -3-\lambda \end{pmatrix}$$

Eigenvector for eigenvalue 1 (multiplicity 1)

In[53]:=
```
MatrixForm[Bmat[eigenvaluesAmat[[1]]]]
```

Out[53]//MatrixForm=

$$\begin{pmatrix} -4 & 6 & -3 \\ -4 & 7 & -4 \\ -4 & 8 & -5 \end{pmatrix}$$

In[54]:=
```
luB1 = LUDecomposition[Bmat[eigenvaluesAmat[[1]]]] // Quiet
```

Out[54]=

{{{-4, 6, -3}, {1, 1, -1}, {1, 2, 0}}, {1, 2, 3}, 0}

In[55]:=
```
MatrixForm[lB1 = LowerTriangularize[luB1[[1]], -1] + IdentityMatrix[3]];
```

In[56]:=
```
MatrixForm[uB1 = UpperTriangularize[luB1[[1]]]]
```

Out[56]//MatrixForm=

$$\begin{pmatrix} -4 & 6 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

In[57]:=
```
sol1 = (Solve[uB1.{X, Y, α} == {0, 0, 0}, {X, Y}][[1]]) // Quiet
```

Out[57]=

$$\left\{X \to \frac{3\,\alpha}{4},\ Y \to \alpha\right\}$$

In[58]:=
```
evec1[α_] = {X, Y, α} /. sol1
```

Out[58]=

$$\left\{\frac{3\,\alpha}{4},\ \alpha,\ \alpha\right\}$$

In[59]:=
```
ev1 = Normalize[evec1[1]]
```
Out[59]=

$$\left\{\frac{3}{\sqrt{41}}, \frac{4}{\sqrt{41}}, \frac{4}{\sqrt{41}}\right\}$$

Eigenvectors for eigenvalue 2 (multiplicity 2)

Here we find two eigenvectors

In[60]:=
```
MatrixForm[Bmat[eigenvaluesAmat[[2]]]]
```
Out[60]//MatrixForm=

$$\begin{pmatrix} -3 & 6 & -3 \\ -4 & 8 & -4 \\ -4 & 8 & -4 \end{pmatrix}$$

In[61]:=
```
luB2 = LUDecomposition[Bmat[eigenvaluesAmat[[2]]]] // Quiet
```
Out[61]=

$$\left\{\left\{\{-3, 6, -3\}, \left\{\frac{4}{3}, 0, 0\right\}, \left\{\frac{4}{3}, 0, 0\right\}\right\}, \{1, 2, 3\}, 0\right\}$$

In[62]:=
```
MatrixForm[lB2 = LowerTriangularize[luB2[[1]], -1] + IdentityMatrix[3]];
```

In[63]:=
```
MatrixForm[uB2 = UpperTriangularize[luB2[[1]]]]
```
Out[63]//MatrixForm=

$$\begin{pmatrix} -3 & 6 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In[64]:=
```
sol2 = (Solve[uB2.{X, β, α} == {0, 0, 0}, {X, Y}][[1]]) // Quiet
```
Out[64]=

$$\{X \to -\alpha + 2\beta\}$$

In[65]:=
```
evec2[α_, β_] = {X, β, α} /. sol2
```
Out[65]=

$$\{-\alpha + 2\beta, \beta, \alpha\}$$

In[66]:=
```
ev21 = Normalize[evec2[1, 0]]
```
Out[66]=

$$\left\{-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}$$

In[67]:=
```
ev22 = Normalize[evec2[0, 1]]
```
Out[67]=

$$\left\{\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right\}$$

Eigenbasis

In[68]:= `MatrixForm[Ebasis = Transpose[{ev1, ev21, ev22}]]`

Out[68]//MatrixForm=

$$\begin{pmatrix} \frac{3}{\sqrt{41}} & -\frac{1}{\sqrt{2}} & \frac{2}{\sqrt{5}} \\ \frac{4}{\sqrt{41}} & 0 & \frac{1}{\sqrt{5}} \\ \frac{4}{\sqrt{41}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

In[69]:= `MatrixForm[G = Transpose[Ebasis].Ebasis // FullSimplify]`

Out[69]//MatrixForm=

$$\begin{pmatrix} 1 & \frac{1}{\sqrt{82}} & 2\sqrt{\frac{5}{41}} \\ \frac{1}{\sqrt{82}} & 1 & -\sqrt{\frac{2}{5}} \\ 2\sqrt{\frac{5}{41}} & -\sqrt{\frac{2}{5}} & 1 \end{pmatrix}$$

Example 4

Eigenvalues

In[70]:= `Clear[Amat]`

In[71]:= `MatrixForm[Amat = A⟦4⟧]`

Out[71]//MatrixForm=

$$\begin{pmatrix} \frac{13}{9} & -\frac{2}{9} & \frac{4}{9} \\ -\frac{2}{9} & \frac{10}{9} & -\frac{2}{9} \\ \frac{4}{9} & -\frac{2}{9} & \frac{13}{9} \end{pmatrix}$$

In[72]:= `CharacteristicPolynomial[Amat, λ] // Factor`

Out[72]=

$$-\left((-2+\lambda)\ (-1+\lambda)^2\right)$$

In[73]:= `eigenvaluesAmat = Eigenvalues[Amat]`

Out[73]=

{2, 1, 1}

In[74]:= `MatrixForm[Bmat[λ_] = Amat - λ * IdentityMatrix[3]]`

Out[74]//MatrixForm=

$$\begin{pmatrix} \frac{13}{9}-\lambda & -\frac{2}{9} & \frac{4}{9} \\ -\frac{2}{9} & \frac{10}{9}-\lambda & -\frac{2}{9} \\ \frac{4}{9} & -\frac{2}{9} & \frac{13}{9}-\lambda \end{pmatrix}$$

Eigenvector for eigenvalue 1 (multiplicity 1)

In[75]:= `MatrixForm[Bmat[eigenvaluesAmat⟦1⟧]]`

Out[75]//MatrixForm=

$$\begin{pmatrix} -\frac{5}{9} & -\frac{2}{9} & \frac{4}{9} \\ -\frac{2}{9} & -\frac{8}{9} & -\frac{2}{9} \\ \frac{4}{9} & -\frac{2}{9} & -\frac{5}{9} \end{pmatrix}$$

In[76]:=
```
luB1 = LUDecomposition[Bmat[eigenvaluesAmat[[1]]]] // Quiet
```
Out[76]=

$$\left\{\left\{\left\{-\frac{2}{9}, -\frac{8}{9}, -\frac{2}{9}\right\}, \left\{\frac{5}{2}, 2, 1\right\}, \{-2, -1, 0\}\right\}, \{2, 1, 3\}, 0\right\}$$

In[77]:=
```
MatrixForm[lB1 = LowerTriangularize[luB1[[1]], -1] + IdentityMatrix[3]];
```
In[78]:=
```
MatrixForm[uB1 = UpperTriangularize[luB1[[1]]]]
```
Out[78]//MatrixForm=

$$\begin{pmatrix} -\frac{2}{9} & -\frac{8}{9} & -\frac{2}{9} \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

In[79]:=
```
sol1 = (Solve[uB1.{X, Y, α} == {0, 0, 0}, {X, Y}][[1]]) // Quiet
```
Out[79]=

$$\left\{X \to \alpha, Y \to -\frac{\alpha}{2}\right\}$$

In[80]:=
```
evec1[α_] = {X, Y, α} /. sol1
```
Out[80]=

$$\left\{\alpha, -\frac{\alpha}{2}, \alpha\right\}$$

In[81]:=
```
ev1 = Normalize[evec1[1]]
```
Out[81]=

$$\left\{\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right\}$$

Eigenvectors for eigenvalue 2 (multiplicity 2)

Here we find two eigenvectors

In[82]:=
```
MatrixForm[Bmat[eigenvaluesAmat[[2]]]]
```
Out[82]//MatrixForm=

$$\begin{pmatrix} \frac{4}{9} & -\frac{2}{9} & \frac{4}{9} \\ -\frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ \frac{4}{9} & -\frac{2}{9} & \frac{4}{9} \end{pmatrix}$$

In[83]:=
```
luB2 = LUDecomposition[Bmat[eigenvaluesAmat[[2]]]] // Quiet
```
Out[83]=

$$\left\{\left\{\left\{-\frac{2}{9}, \frac{1}{9}, -\frac{2}{9}\right\}, \{-2, 0, 0\}, \{-2, 0, 0\}\right\}, \{2, 1, 3\}, 0\right\}$$

In[84]:=
```
MatrixForm[lB2 = LowerTriangularize[luB2[[1]], -1] + IdentityMatrix[3]];
```
In[85]:=
```
MatrixForm[uB2 = UpperTriangularize[luB2[[1]]]]
```
Out[85]//MatrixForm=

$$\begin{pmatrix} -\frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In[86]:=
```
sol2 = (Solve[uB2.{X, β, α} == {0, 0, 0}, {X, Y}][[1]]) // Quiet
```

Out[86]=

$$\left\{X \to -\alpha + \frac{\beta}{2}\right\}$$

In[87]:=
```
evec2[α_, β_] = {X, β, α} /. sol2
```

Out[87]=

$$\left\{-\alpha + \frac{\beta}{2}, \beta, \alpha\right\}$$

In[88]:=
```
ev21 = Normalize[evec2[1, 0]]
```

Out[88]=

$$\left\{-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}$$

In[89]:=
```
ev22 = Normalize[evec2[0, 1]]
```

Out[89]=

$$\left\{\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right\}$$

Eigenbasis

In[90]:=
```
MatrixForm[Ebasis = Transpose[{ev1, ev21, ev22}]]
```

Out[90]//MatrixForm=

$$\begin{pmatrix} \frac{2}{3} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{3} & 0 & \frac{2}{\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

In[91]:=
```
MatrixForm[G = Transpose[Ebasis].Ebasis // FullSimplify]
```

Out[91]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{\sqrt{10}} \\ 0 & -\frac{1}{\sqrt{10}} & 1 \end{pmatrix}$$

The two eigenvectors for the second eigenvalues can be orthogonalized, such that the basis of eigenvectors become orthonormal:

In[92]:=
```
aux = Orthogonalize[{ev21, ev22}]
```

Out[92]=

$$\left\{\left\{-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{3\sqrt{2}}, \frac{2\sqrt{2}}{3}, \frac{1}{3\sqrt{2}}\right\}\right\}$$

In[93]:=
```
ev21prime = aux[[1]]
```

Out[93]=

$$\left\{-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}$$

In[94]:=
```
ev22prime = aux[[2]]
```
Out[94]=

$$\left\{\frac{1}{3\sqrt{2}}, \frac{2\sqrt{2}}{3}, \frac{1}{3\sqrt{2}}\right\}$$

In[95]:=
```
MatrixForm[EbasisPrime = Transpose[{ev1, ev21prime, ev22prime}]]
```
Out[95]//MatrixForm=

$$\begin{pmatrix} \frac{2}{3} & -\frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ -\frac{1}{3} & 0 & \frac{2\sqrt{2}}{3} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \end{pmatrix}$$

In[96]:=
```
MatrixForm[Gprime = Transpose[EbasisPrime].EbasisPrime // FullSimplify]
```
Out[96]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 5

Eigenvalues

In[97]:=
```
Clear[Amat]
```

In[98]:=
```
MatrixForm[Amat = A[[5]]]
```
Out[98]//MatrixForm=

$$\begin{pmatrix} 2 & 1 & -1 \\ 4 & -1 & 0 \\ 8 & -7 & 3 \end{pmatrix}$$

In[99]:=
```
CharacteristicPolynomial[Amat, λ] // Factor
```
Out[99]=

$$-\left((-2+\lambda)\ (-1+\lambda)^2\right)$$

In[100]:=
```
eigenvaluesAmat = Eigenvalues[Amat]
```
Out[100]=

$$\{2, 1, 1\}$$

In[101]:=
```
MatrixForm[Bmat[λ_] = Amat - λ * IdentityMatrix[3]]
```
Out[101]//MatrixForm=

$$\begin{pmatrix} 2-\lambda & 1 & -1 \\ 4 & -1-\lambda & 0 \\ 8 & -7 & 3-\lambda \end{pmatrix}$$

Eigenvector for eigenvalue 1 (multiplicity 1)

In[102]:=
```
MatrixForm[Bmat[eigenvaluesAmat[[1]]]]
```
Out[102]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & -1 \\ 4 & -3 & 0 \\ 8 & -7 & 1 \end{pmatrix}$$

In[103]:=

```
luB1 = LUDecomposition[Bmat[eigenvaluesAmat⟦1⟧]] // Quiet
```

Out[103]=

{{{4, -3, 0}, {0, 1, -1}, {2, -1, 0}}, {2, 1, 3}, 0}

In[104]:=

```
MatrixForm[lB1 = LowerTriangularize[luB1⟦1⟧, -1] + IdentityMatrix[3]];
```

In[105]:=

```
MatrixForm[uB1 = UpperTriangularize[luB1⟦1⟧]]
```

Out[105]//MatrixForm=

$$\begin{pmatrix} 4 & -3 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

In[106]:=

```
sol1 = (Solve[uB1.{X, Y, α} == {0, 0, 0}, {X, Y}]⟦1⟧) // Quiet
```

Out[106]=

$$\left\{X \to \frac{3\,\alpha}{4},\ Y \to \alpha\right\}$$

In[107]:=

```
evec1[α_] = {X, Y, α} /. sol1
```

Out[107]=

$$\left\{\frac{3\,\alpha}{4},\ \alpha,\ \alpha\right\}$$

In[108]:=

```
ev1 = Normalize[evec1[1]]
```

Out[108]=

$$\left\{\frac{3}{\sqrt{41}},\ \frac{4}{\sqrt{41}},\ \frac{4}{\sqrt{41}}\right\}$$

Eigenvectors for eigenvalue 2 (multiplicity 2)

Here we find only one eigenvector

In[109]:=

```
MatrixForm[Bmat[eigenvaluesAmat⟦2⟧]]
```

Out[109]//MatrixForm=

$$\begin{pmatrix} 1 & 1 & -1 \\ 4 & -2 & 0 \\ 8 & -7 & 2 \end{pmatrix}$$

In[110]:=

```
luB2 = LUDecomposition[Bmat[eigenvaluesAmat⟦2⟧]] // Quiet
```

Out[110]=

$$\left\{\left\{\{1, 1, -1\}, \{4, -6, 4\}, \left\{8, \frac{5}{2}, 0\right\}\right\}, \{1, 2, 3\}, 0\right\}$$

In[111]:=

```
MatrixForm[lB2 = LowerTriangularize[luB2⟦1⟧, -1] + IdentityMatrix[3]];
```

In[112]:=

```
MatrixForm[uB2 = UpperTriangularize[luB2[[1]]]]
```

Out[112]//MatrixForm=

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & -6 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

In[113]:=

```
sol2 = (Solve[uB2.{X, Y, α} == {0, 0, 0}, {X, Y}][[1]]) // Quiet
```

Out[113]=

$$\left\{X \to \frac{\alpha}{3},\ Y \to \frac{2\,\alpha}{3}\right\}$$

In[114]:=

```
evec2[α_] = {X, Y, α} /. sol2
```

Out[114]=

$$\left\{\frac{\alpha}{3},\ \frac{2\,\alpha}{3},\ \alpha\right\}$$

In[115]:=

```
ev21 = Normalize[evec2[1]]
```

Out[115]=

$$\left\{\frac{1}{\sqrt{14}},\ \sqrt{\frac{2}{7}},\ \frac{3}{\sqrt{14}}\right\}$$

Reduced eigenbasis

In[116]:=

```
MatrixForm[Ebasis = Transpose[{ev1, ev21}]]
```

Out[116]//MatrixForm=

$$\begin{pmatrix} \frac{3}{\sqrt{41}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{41}} & \sqrt{\frac{2}{7}} \\ \frac{4}{\sqrt{41}} & \frac{3}{\sqrt{14}} \end{pmatrix}$$

In[117]:=

```
MatrixForm[G = Transpose[Ebasis].Ebasis // FullSimplify]
```

Out[117]//MatrixForm=

$$\begin{pmatrix} 1 & \frac{23}{\sqrt{574}} \\ \frac{23}{\sqrt{574}} & 1 \end{pmatrix}$$